

# FINSLER METRICS OF CONSTANT POSITIVE CURVATURE ON THE LIE GROUP $S^3$

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**ABSTRACT.** Guided by the Hopf fibration, we single out a family (indexed by a positive constant  $K$ ) of right invariant Riemannian metrics on the Lie group  $S^3$ . Using the Yasuda–Shimada theorem as an inspiration, we determine for each  $K > 1$  a privileged right invariant Killing field of constant length. Each such Riemannian metric pairs with the corresponding Killing field to produce a  $y$ -global and *explicit* Randers metric on  $S^3$ . Using the machinery of spray curvature and Berwald’s formula for it, we prove directly that the said Randers metric has constant positive flag curvature  $K$ , as predicted by the Yasuda–Shimada theorem. We also explain why this family of Finslerian space forms is *not* projectively flat.

## 1. INTRODUCTION

A Finsler metric  $F$  is a family of ‘norms’ on a manifold  $M$ , one on each tangent space  $T_x M$ . These ‘norms’ are typically only positively homogeneous of degree one, whereas the norms used in functional analysis are absolutely homogeneous. There are also the usual smoothness and strong convexity assumptions (see for instance [BCS]) on the slit tangent bundle  $TM \setminus 0$ . In a large number of examples, especially the ones of physical origin, these technical requirements are only satisfied on open cones in  $TM \setminus 0$ . If the Finsler metric is smooth and strongly convex on the *entire* slit tangent bundle  $TM \setminus 0$ , it is said to be  $y$ -global. Examples that are both  $y$ -global and geometrically significant are highly sought after.

In Riemannian geometry, one has the concept of sectional curvature. Its analogue in Finsler geometry is called the flag curvature. Flag curvatures are more easily accessed in some settings through spray curvatures. These objects have now been given detailed treatments in textbooks and monographs such as [R], [BCS], [AIM], and [S1]. Finsler spaces of constant flag curvature are, just like their Riemannian counterparts, known as space forms. However, unlike Riemannian geometry, Finslerian space forms do

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not just arise from three standard models. This phenomenon is related to the fact that on  $\mathbb{R}^n$ , there are many isometry classes of norms, but only one isometry class of inner products.

Akbar-Zadeh [AZ] showed that if the metrics in question are geodesically complete and the growth of their Cartan tensors are suitably constrained, then flat and negatively curved space forms are fairly well understood. See [BCS] for a leisurely exposition. Relax any of those two hypotheses and one encounters intriguing spaces, like for example the Finslerian Poincaré disc discussed in [BCS].

This paper deals with  $y$ -global Finsler spaces of constant positive flag curvature, and puts us in an even more esoteric landscape. In this regard, we have several pioneering works of Bryant's. Two dimensional examples are treated in [Br1], [Br2], [Br3]; higher dimensional ones are discussed in his lectures. As far as we know, all of Bryant's examples are projectively flat (meaning that their geodesics are straight lines in certain coordinate systems), and none of them is of the Randers vintage.

Randers spaces are Finsler spaces constructed from just two pieces of familiar data: a Riemannian metric and a differential 1-form, both globally defined on an underlying smooth manifold. As such, they are possibly the best stepping stones from the Riemannian realm to the Finslerian territory.

The goal of this paper is to produce, for each constant  $K > 1$ , an explicit example of a compact boundaryless (non-Riemannian) Randers space that has constant positive flag curvature  $K$ , and which is not projectively flat.

For that purpose, we turn to the Yasuda–Shimada theorem [YS]. This result was published in the late 70s, and classifies Randers metrics of constant positive and constant negative flag curvature. Incidentally, flat Randers metrics are necessarily locally Minkowskian. For the positively curved case, the Yasuda–Shimada theorem gives four mathematical criteria on the Riemannian metric and the 1-form. These criteria are made precise in §2.

For each  $K > 1$ , we select a manifold  $M$  judiciously, then solve those four criteria for a Riemannian metric  $\tilde{a}$  and a nonzero 1-form  $\tilde{b}$ , both living on  $M$ . This is carried out in §3–§5. By doing so, we have shown that the Yasuda–Shimada criteria are non-vacuous. The simplest choice of  $M$  turns out to be the Lie group  $S^3$ . Similar promise [Ro] holds for  $S^{2n+1}$ , mainly because every odd-dimensional sphere admits the Hopf fibration.

After the construction, we could in principle appeal to the Yasuda–Shimada theorem to conclude that our Randers spaces have constant positive flag curvature. However, the two published proofs ([YS] and [M]) of the theorem in question both contain arguments that we find difficult to

follow. In order to render our paper logically self-contained, we prove directly that each said example indeed has constant positive flag curvature  $K$ . The method we use is quite different from the ones in [YS] and [M]. It is geometrical and brings out the spectacular power behind a formula of Berwald's. It also reaffirms our belief that a happy synergy awaits Finsler geometry and modern computing.

To that end, we first review the concept of spray curvatures, and its close relationship with the flag curvature, in §6. We then give Berwald's formula for the spray curvature. This formula has proved to be exceptionally useful in computational Finsler geometry, whether one carries out the computations on machine or by hand.

In the Appendix, we write a Maple program based on Berwald's formula and use it to generate numerical data, supporting the contention that our Randers metrics have constant positive flag curvature  $K$ . This type of experimental evidence provides the faith which sustains our direct *proof* (by hand) in §7. The hand computations in this direct proof are facilitated by a refined understanding [S1] of the spray curvature, when the latter is adapted to the Randers setting.

We believe that this refined understanding also holds the key to a distinctly new proof of the Yasuda–Shimada theorem. This theorem has been a reliable source of beautiful examples. For instance, the Finslerian Poincaré disc we cited above has a Yasuda–Shimada pedigree as well. Given that, it is desirable to have a more accessible and *geometric* proof of the theorem. The outcome of such an endeavor will be reported elsewhere.

Finally, in §8, we give a rather preliminary discussion on the geodesics of our Randers metrics. Specifically, we invoke standard results about the projective Weyl and Douglas tensors to deduce that there is no coordinate system in which the geodesics appear as straight lines. Thus, as promised, our Randers spaces are not projectively flat.

## 2. RANDERS SPACES AND THE YASUDA–SHIMADA THEOREM

Randers metrics were introduced by Randers in 1941 [Ra] in the context of general relativity. They play a prominent role in Ingarden's study of electron optics (see his treatment of the subject in [AIM]). Mathematically, in spite of the wide range of non-Riemannian phenomena they are capable of producing, Randers spaces are Finsler spaces built from data that are quite familiar to all differential geometers:

- a Riemannian metric  $\tilde{a} := \tilde{a}_{ij} dx^i \otimes dx^j$  on a smooth  $n$ -dimensional manifold  $M$ , and
- a differential 1-form  $\tilde{b} := \tilde{b}_i dx^i$  on  $M$ . This  $\tilde{b}$  is sometimes called a drift 1-form.

Together these objects define a Finsler metric  $F$  in a simple way:

$$F(x, y) := \alpha(x, y) + \beta(x, y),$$

where

$$\begin{aligned}\alpha(x, y) &:= \sqrt{\tilde{a}_{ij}(x) y^i y^j} \\ \beta(x, y) &:= \tilde{b}_i(x) y^i.\end{aligned}$$

Here,  $x$  stands for points on the manifold  $M$ , and  $y \in T_x M$  denote tangent vectors based at  $x$ . Those tangent space coordinates  $y^i$  typically come from the expansion  $y = y^i \frac{\partial}{\partial x^i}$  in terms of a local coordinate basis. They can also arise, as  $y^p$ , from the expansion  $y = y^p e_p$  in terms of an  $\tilde{a}$ -orthonormal frame field on  $M$ . Both scenarios are exemplified in §5.

Due to the presence of  $\beta$ , the Randers metric  $F := \alpha + \beta$  is generally only positively homogeneous of degree one in  $y$ :  $F(x, cy) = cF(x, y)$  for all positive  $c$ . A Randers metric cannot be absolutely homogeneous [ $F(x, cy) = |c|F(x, y)$ ] unless  $\tilde{b} = 0$ , in which case  $F$  is Riemannian.

The fundamental tensor is formally analogous to the metric tensor in Riemannian geometry. It is defined as

$$g_{ij} := \frac{1}{2} (F^2)_{y^i y^j},$$

where we have used  $y^i$ ,  $y^j$  as subscripts to signify partial differentiation. Almost by inspection, we have

$$\begin{aligned}\tilde{\ell}_i &:= \alpha_{y^i} = \frac{\tilde{a}_{ij} y^j}{\alpha}, \\ \ell_i &:= F_{y^i} = \tilde{\ell}_i + \tilde{b}_i.\end{aligned}$$

The fundamental tensor can then be expressed as

$$g_{ij} = \frac{F}{\alpha} \left( \tilde{a}_{ij} - \tilde{\ell}_i \tilde{\ell}_j \right) + \ell_i \ell_j.$$

Since  $\beta(x, y)$  is linear in  $y$ , it cannot possibly have a fixed sign. The size of  $\tilde{b}$  therefore needs to be controlled if  $F$  is to be positive on  $TM \setminus 0$ . We also want the fundamental tensor to be positive definite. It turns out that both these properties hold if and only if

$$\|\tilde{b}\| := \sqrt{\tilde{b}_i \tilde{b}^i} < 1,$$

where

$$\tilde{b}^i := \tilde{a}^{ij} \tilde{b}_j.$$

So, the drift 1-form  $\tilde{b}$  of Randers spaces must be required to have Riemannian norm strictly smaller than 1 everywhere. See [BCS] or [AIM].

The notion of flag curvature makes sense for all Finsler spaces. It is constructed from the  $hh$ -curvature tensor of one's favorite Finsler connection. Happily, the resulting flag curvature is independent of which standard connection one is using, be it Berwald's, Cartan's, Chern's, or Hashiguchi's, just to name a few. Flag and spray curvatures will be reviewed in §6 rather than here, in order to effect a more streamlined exposition.

The Yasuda–Shimada theorem [YS] says that for a Randers metric to have constant *positive* flag curvature  $K$ , the following four criteria are both necessary and sufficient:

- The 1-form  $\tilde{b}$  is a Killing field of the Riemannian metric  $\tilde{a}$ .

$$\tilde{b}_{i|j} + \tilde{b}_{j|i} = 0.$$

- The Riemannian norm of this Killing field must be constant, besides being globally less than 1.

$$1 > \|\tilde{b}\|^2 := \tilde{b}_i \tilde{b}^i \text{ is constant.}$$

- Second order covariant derivatives of  $\tilde{b}$  are to have the specific form

$$\tilde{b}_{i|j|k} = K \left( \tilde{a}_{ik} \tilde{b}_j - \tilde{a}_{jk} \tilde{b}_i \right).$$

- The Riemann curvature tensor  $\tilde{R}_{hijk}$  of the metric  $\tilde{a}$  must be given by the special formula

$$\begin{aligned} & - K \left( 1 - \|\tilde{b}\|^2 \right) \tilde{a}_{hj} \tilde{a}_{ik} - K \left( \tilde{a}_{hj} \tilde{b}_i \tilde{b}_k + \tilde{a}_{ik} \tilde{b}_h \tilde{b}_j \right) + \tilde{b}_{h|j} \tilde{b}_{i|k} \\ & + K \left( 1 - \|\tilde{b}\|^2 \right) \tilde{a}_{hk} \tilde{a}_{ij} + K \left( \tilde{a}_{hk} \tilde{b}_i \tilde{b}_j + \tilde{a}_{ij} \tilde{b}_h \tilde{b}_k \right) - \tilde{b}_{h|k} \tilde{b}_{i|j} \\ & + 2 \tilde{b}_{h|i} \tilde{b}_{j|k}. \end{aligned}$$

Here, the vertical slash  $(\dots)_{i|j}$  denotes covariant differentiation on  $M$ , taken with respect to the Levi-Civita (Christoffel) connection of the Riemannian metric  $\tilde{a}$ .

In order to obtain a non-Riemannian Randers metric from these four criteria, the drift 1-form needs to be nowhere zero because it has constant length. Limiting our search to compact oriented manifolds  $M$  without boundary, we deduce from the Poincaré–Hopf index theorem that the Euler characteristic  $\chi(M)$  must vanish.

In two dimensions, the only candidate for  $M$  is therefore the torus. On this manifold, the most easily visualized Riemannian metric  $\tilde{a}$  is the one that gives the torus of revolution in  $\mathbb{R}^3$ . In that case, a straightforward calculation shows that every constant length Killing field  $\tilde{b}$  is identically zero. So there is no non-Riemannian Randers metric with constant positive flag curvature on the torus of revolution.

It is not obvious whether the above conclusion again results if we use other Riemannian metrics  $\tilde{a}$  on the torus. If we are in the boundaryless category, there are telltale indications ([S1], [S2]) that complete, positively homogeneous [as opposed to the more restrictive absolute homogeneity  $F(x, cy) = |c|F(x, y)$ ] Finsler metrics of constant positive flag curvature can only be supported on manifolds homeomorphic to spheres.

### 3. THE LIE GROUP $S^3$ AND ITS HOPF FIBRATION

Our goal is to find a non-Riemannian Randers metric by ‘solving’ the four criteria stated in the Yasuda–Shimada theorem. This means we must produce a manifold  $M$  and, living globally on it, a special Riemannian metric  $\tilde{a}$  and a nonzero differential 1-form  $\tilde{b}$  that is appropriately coupled to  $\tilde{a}$ . As explained near the end of §2, the base manifold  $M$  must have zero Euler characteristic. Working in odd dimensions and staying in the boundaryless category automatically satisfies this topological constraint.

The simplest compact boundaryless oriented 3-manifold is the Lie group  $S^3$ , which is also a circle bundle over  $S^2$ . We briefly discuss these properties for the sole purpose of setting some notation.

- As a Riemannian manifold,  $S^3$  is the standard unit sphere in Euclidean  $\mathbb{R}^4$ , whose points  $x$  have Cartesian coordinates  $(x^0, x^1, x^2, x^3)$ . In other words,  $S^3$  can be characterized by

$$|x| := \sqrt{(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2} = 1.$$

- As a Lie group,  $S^3$  consists of the unit quaternions

$$x = x^0 \mathbf{1} + x^1 \mathbf{i} + x^2 \mathbf{j} + x^3 \mathbf{k}$$

in the space  $\mathbb{H}$  of quaternions. Its group structure is isomorphic to  $SU(2)$ . Multiplication between quaternions is non-commutative and is governed by the formal rules

$$\begin{aligned} \mathbf{ii} &= -\mathbf{1}, & \mathbf{jj} &= -\mathbf{1}, & \mathbf{kk} &= -\mathbf{1}, \\ \mathbf{ij} &= +\mathbf{k}, & \mathbf{jk} &= +\mathbf{i}, & \mathbf{ki} &= +\mathbf{j}, \\ \mathbf{ji} &= -\mathbf{ij}, & \mathbf{kj} &= -\mathbf{jk}, & \mathbf{ik} &= -\mathbf{ki}, \end{aligned}$$

where the object  $\mathbf{1}$  acts just like the real number 1. In view of the remarkable formula  $|x\bar{x}| = |x||\bar{x}|$ , the unit quaternions are closed under the defined multiplication. Also,  $x\bar{x} = |x|^2 = \bar{x}x$ , with

$$\bar{x} := x^0 \mathbf{1} - x^1 \mathbf{i} - x^2 \mathbf{j} - x^3 \mathbf{k}.$$

So the group inverse of any  $x \in S^3$  is its conjugate:

$$x^{-1} = \bar{x}.$$

As a circle bundle,  $S^3$  has the celebrated Hopf fibration. For that purpose, it is best described as the set of

$$(z, w) := (x^o + ix^1, x^2 + ix^3)$$

in  $\mathbb{C} \times \mathbb{C}$  satisfying  $|z|^2 + |w|^2 = 1$ . Incidentally, with this notation, the aforementioned isomorphism between  $S^3$  and  $SU(2)$  reads

$$(z, w) \leftrightarrow \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}.$$

The unit complex numbers  $e^{i\theta} \in S^1$  act on  $S^3$  by

$$(z, w) \mapsto (e^{i\theta}z, e^{i\theta}w).$$

The quotient manifold is the same as  $\mathbb{C}^2$  mod first the length and then the argument of an arbitrary complex number. So it is  $\mathbb{C}P^1$  which, by a suitable stereographic projection (see [F]), is precisely the complex manifold  $S^2$ . Thus  $S^3$  is a circle bundle over  $S^2$ . This bundle is non-trivial because  $S^3$  is simply connected, whereas  $S^2 \times S^1$  is not.

Using a curve of angles  $\theta(t)$  with  $\theta(0) = 0$ , one calculates the infinitesimal generator

$$\frac{d}{dt}_{|t=0} (e^{i\theta(t)}z, e^{i\theta(t)}w)$$

of the said  $S^1$  action. The answer is  $\theta'(0)$  times the following distinguished vector field

$$E_1 := -x^1\partial_o + x^o\partial_1 - x^3\partial_2 + x^2\partial_3,$$

where  $\partial_\mu$  abbreviates  $\partial_{x^\mu}$ . This  $E_1$  is therefore tangent to the  $S^1$  fibres. It is complemented by

$$E_2 := -x^2\partial_o + x^3\partial_1 + x^o\partial_2 - x^1\partial_3$$

$$E_3 := -x^3\partial_o - x^2\partial_1 + x^1\partial_2 + x^o\partial_3$$

to give a global orthonormal frame field on  $S^3$  with

$$[E_1, E_2] = -2E_3, \quad [E_2, E_3] = -2E_1, \quad [E_3, E_1] = -2E_2.$$

At any point  $x \in S^3$ , the values of  $E_1, E_2, E_3$  are respectively equal to the quaternion products  $ix, jx$ , and  $kx$ . This fact can be used to check that our frame field is right invariant.

The natural dual of  $\{E_1, E_2, E_3\}$  is the following globally defined right invariant orthonormal coframe on  $S^3$ :

$$\Theta^1 := -x^1dx^o + x^o dx^1 - x^3 dx^2 + x^2 dx^3,$$

$$\Theta^2 := -x^2dx^o + x^3 dx^1 + x^o dx^2 - x^1 dx^3,$$

$$\Theta^3 := -x^3dx^o - x^2 dx^1 + x^1 dx^2 + x^o dx^3.$$

The standard metric on  $S^3$  is thus  $\Theta^1 \otimes \Theta^1 + \Theta^2 \otimes \Theta^2 + \Theta^3 \otimes \Theta^3$ . Also, one has

$$d\Theta^1 = 2\Theta^2 \wedge \Theta^3, \quad d\Theta^2 = 2\Theta^3 \wedge \Theta^1, \quad d\Theta^3 = 2\Theta^1 \wedge \Theta^2.$$

Motivated by the treatment of the Hopf fibration in [GLP], we consider the following Riemannian metric on  $S^3$ :

$$\tilde{a} := \epsilon^2 \Theta^1 \otimes \Theta^1 + \Theta^2 \otimes \Theta^2 + \Theta^3 \otimes \Theta^3,$$

where  $\epsilon$  is a positive constant (typically different from 1). This  $\tilde{a}$  modifies the standard metric on  $S^3$  by introducing a dilation along the  $S^1$  fibres. The Lie derivative of  $\tilde{a}$  can be calculated by using the Cartan formula  $\mathcal{L}_X\Theta = i_X d\Theta + di_X\Theta$  on  $\Theta^1$ ,  $\Theta^2$ , and  $\Theta^3$ . We find that every constant multiple of  $E_1$  is a Killing vector field of  $\tilde{a}$ , whereas  $E_2$ ,  $E_3$  are not Killing fields unless  $\epsilon$  happens to be 1.

The Riemannian metric  $\tilde{a}$  admits its own orthonormal frame field

$$e_1 := \frac{1}{\epsilon} E_1, \quad e_2 := E_2, \quad e_3 := E_3.$$

The natural dual consists of

$$\omega^1 := \epsilon \Theta^1, \quad \omega^2 := \Theta^2, \quad \omega^3 := \Theta^3,$$

with

$$d\omega^1 = 2\epsilon \omega^2 \wedge \omega^3, \quad d\omega^2 = \frac{2}{\epsilon} \omega^3 \wedge \omega^1, \quad d\omega^3 = \frac{2}{\epsilon} \omega^1 \wedge \omega^2.$$

Relative to this  $\tilde{a}$ -orthonormal frame field, the Levi-Civita (Christoffel) connection of  $\tilde{a}$  consists of a skew-symmetric  $3 \times 3$  matrix of 1-forms  $\omega_q^p$ . These are obtained by solving the structural equations  $d\omega^p = \omega^q \wedge \omega_q^p$  and  $\omega_{pq} = -\omega_{qp}$ . Here,  $\omega_{qp}$  means  $\omega_q^s \delta_{sp}$ , which is numerically the same as  $\omega_q^p$ . We find that

$$\begin{pmatrix} \omega_1^1 & \omega_1^2 & \omega_1^3 \\ \omega_2^1 & \omega_2^2 & \omega_2^3 \\ \omega_3^1 & \omega_3^2 & \omega_3^3 \end{pmatrix} = \begin{pmatrix} 0 & -\epsilon \omega^3 & \epsilon \omega^2 \\ \epsilon \omega^3 & 0 & (\epsilon - \frac{2}{\epsilon}) \omega^1 \\ -\epsilon \omega^2 & (\frac{2}{\epsilon} - \epsilon) \omega^1 & 0 \end{pmatrix}.$$

The curvature 2-forms and the Riemann curvature tensor are related by

$$d\omega_q^p - \omega_q^s \wedge \omega_s^p = \frac{1}{2} \tilde{R}_q^p{}_{rs} \omega^r \wedge \omega^s,$$

where our convention for indices on  $\tilde{R}_q^p{}_{rs}$  follows that in [BCS]. Since we are in an orthonormal frame,  $\tilde{R}_q^p{}_{rs}$  is numerically equal to  $\tilde{R}_{qprs}$ . Computations give

$$\begin{aligned} \tilde{R}_{1212} &= -\epsilon^2, & \tilde{R}_{1213} &= 0, & \tilde{R}_{1223} &= 0, \\ \tilde{R}_{1312} &= 0, & \tilde{R}_{1313} &= -\epsilon^2, & \tilde{R}_{1323} &= 0, \\ \tilde{R}_{2312} &= 0, & \tilde{R}_{2313} &= 0, & \tilde{R}_{2323} &= 3\epsilon^2 - 4. \end{aligned}$$

All other components of the Riemann curvature of  $\tilde{a}$  are obtained from these by standard properties. Namely,  $\tilde{R}_{qpsr} = -\tilde{R}_{qprs}$ ,  $\tilde{R}_{pqrs} = -\tilde{R}_{qprs}$ , and the block symmetry  $\tilde{R}_{rsqp} = \tilde{R}_{qprs}$ .

#### 4. A PRIVILEGED KILLING FIELD ON $S^3$

In §3, we discussed the Hopf fibration of  $S^3$  and introduced the vector field  $E_1$  that is tangent to the  $S^1$  fibres. The natural dual of this  $E_1$  is the 1-form  $\Theta^1$ , also explicitly presented in §3. We pointed out that every constant multiple of  $E_1$  is a Killing vector (equivalently, every constant multiple of  $\Theta^1$  is a Killing covector) field of the Riemannian metric

$$\begin{aligned}\tilde{a} &:= \epsilon^2 \Theta^1 \otimes \Theta^1 + \Theta^2 \otimes \Theta^2 + \Theta^3 \otimes \Theta^3 \\ &= \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3.\end{aligned}$$

Motivated by this fact, let us stipulate the drift 1-form to be

$$\tilde{b} := \lambda \Theta^1 = \frac{\lambda}{\epsilon} \omega^1,$$

where  $\lambda$  is at the moment an arbitrary but nonzero constant. With respect to the metric  $\tilde{a}$ , this  $\tilde{b}$  has constant Riemannian length

$$\|\tilde{b}\| = \frac{|\lambda|}{\epsilon}.$$

So the pair  $\tilde{a}$  and  $\tilde{b}$  do satisfy the first two criteria (among four) given by the Yasuda–Shimada theorem. Recall that those four criteria are necessary and sufficient for the Randers metric with data  $\tilde{a}$ ,  $\tilde{b}$  to have constant positive flag curvature  $K$ . The purpose of this section is to use the remaining two criteria to determine the constants  $\epsilon$  and  $\lambda$  in terms of  $K$ .

We carry out our calculations in the moving frame  $\{e_1, e_2, e_3\}$  and moving coframe  $\{\omega^1, \omega^2, \omega^3\}$ , which are  $\tilde{a}$ -orthonormal. The connection forms  $\omega_q^p$  are displayed in §3. We have

$$\tilde{b}_1 = \frac{\lambda}{\epsilon}, \quad \tilde{b}_2 = 0, \quad \tilde{b}_3 = 0.$$

And the relevant covariant differentiation formulas are:

$$\begin{aligned}\tilde{b}_{p|q} &= \left( d\tilde{b}_p - \tilde{b}_s \omega_p^s \right) (e_q), \\ \tilde{b}_{p|q|r} &= \left( d\tilde{b}_{p|q} - \tilde{b}_{s|q} \omega_p^s - \tilde{b}_{p|s} \omega_q^s \right) (e_r).\end{aligned}$$

Note that  $\tilde{b}_{p|q}$  is skew-symmetric on its indices because  $\tilde{b}$  is Killing. Consequently, the second covariant derivative  $\tilde{b}_{p|q|r}$  is skew-symmetric in the

indices  $p$  and  $q$ . Straightforward calculations give:

$$\begin{pmatrix} \tilde{b}_{1|1} & \tilde{b}_{1|2} & \tilde{b}_{1|3} \\ \tilde{b}_{2|1} & \tilde{b}_{2|2} & \tilde{b}_{2|3} \\ \tilde{b}_{3|1} & \tilde{b}_{3|2} & \tilde{b}_{3|3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda \\ 0 & +\lambda & 0 \end{pmatrix},$$

and

$$\begin{aligned} \tilde{b}_{1|2|1} &= 0, & \tilde{b}_{1|3|1} &= 0, & \tilde{b}_{2|3|1} &= 0, \\ \tilde{b}_{1|2|2} &= -\lambda\epsilon, & \tilde{b}_{1|3|2} &= 0, & \tilde{b}_{2|3|2} &= 0, \\ \tilde{b}_{1|2|3} &= 0, & \tilde{b}_{1|3|3} &= -\lambda\epsilon, & \tilde{b}_{2|3|3} &= 0. \end{aligned}$$

Recall the third criterion given by Yasuda–Shimada. It reads

$$\tilde{b}_{p|q|r} = T_{pqr}, \quad \text{where } T_{pqr} := K \left( \tilde{a}_{pr} \tilde{b}_q - \tilde{a}_{qr} \tilde{b}_p \right).$$

We find that

$$\begin{aligned} T_{121} &= 0, & T_{131} &= 0, & T_{231} &= 0, \\ T_{122} &= -\frac{\lambda K}{\epsilon}, & T_{132} &= 0, & T_{232} &= 0, \\ T_{123} &= 0, & T_{133} &= -\frac{\lambda K}{\epsilon}, & T_{233} &= 0. \end{aligned}$$

Since the constant  $\lambda$  is nonzero, the third Yasuda–Shimada criterion holds if and only if the dilation factor  $\epsilon$  is given by

$$\epsilon = \sqrt{K}.$$

We have just determined that, for our purpose, the Riemannian metric  $\tilde{a}$  on  $S^3$  should be

$$\begin{aligned} \tilde{a} &:= K \Theta^1 \otimes \Theta^1 + \Theta^2 \otimes \Theta^2 + \Theta^3 \otimes \Theta^3 \\ &= \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3. \end{aligned}$$

This information updates the Riemann curvature tensor we calculated (at the end of §3) to

$$\begin{aligned} \tilde{R}_{1212} &= -K, & \tilde{R}_{1213} &= 0, & \tilde{R}_{1223} &= 0, \\ \tilde{R}_{1312} &= 0, & \tilde{R}_{1313} &= -K, & \tilde{R}_{1323} &= 0, \\ \tilde{R}_{2312} &= 0, & \tilde{R}_{2313} &= 0, & \tilde{R}_{2323} &= 3K - 4. \end{aligned}$$

Now examine the fourth (and last) criterion in the Yasuda–Shimada theorem. It requires the coupling between the drift 1-form  $\tilde{b}$  and the Riemannian metric  $\tilde{a}$  to be such that

$$\tilde{R}_{qprs} = \mathcal{T}_{qprs},$$

where  $\mathcal{T}_{qprs}$  abbreviates the expression

$$\begin{aligned} & -K \left( 1 - \|\tilde{b}\|^2 \right) \tilde{a}_{qr} \tilde{a}_{ps} - K \left( \tilde{a}_{qr} \tilde{b}_p \tilde{b}_s + \tilde{a}_{ps} \tilde{b}_q \tilde{b}_r \right) + \tilde{b}_{q|r} \tilde{b}_{p|s} \\ & + K \left( 1 - \|\tilde{b}\|^2 \right) \tilde{a}_{qs} \tilde{a}_{pr} + K \left( \tilde{a}_{qs} \tilde{b}_p \tilde{b}_r + \tilde{a}_{pr} \tilde{b}_q \tilde{b}_s \right) - \tilde{b}_{q|s} \tilde{b}_{p|r} \\ & + 2 \tilde{b}_{q|p} \tilde{b}_{r|s}. \end{aligned}$$

Calculations give

$$\begin{aligned} \mathcal{T}_{1212} &= -K, & \mathcal{T}_{1213} &= 0, & \mathcal{T}_{1223} &= 0, \\ \mathcal{T}_{1312} &= 0, & \mathcal{T}_{1313} &= -K, & \mathcal{T}_{1323} &= 0, \\ \mathcal{T}_{2312} &= 0, & \mathcal{T}_{2313} &= 0, & \mathcal{T}_{2323} &= 4\lambda^2 - K. \end{aligned}$$

Therefore the fourth criterion of Yasuda–Shimada is satisfied if and only if

$$\lambda = \pm \sqrt{K-1} =: \mathfrak{s}\sqrt{K-1}.$$

In particular, the constant positive flag curvature  $K$  that one is striving for must be  $\geq 1$ . Anyway, the drift 1-form has now been determined. It is

$$\tilde{b} := \pm \sqrt{K-1} \Theta^1 = \pm \sqrt{\frac{K-1}{K}} \omega^1.$$

It is somewhat amazing that the scaling multiple  $\lambda$  (on  $\tilde{b}$ ) asserts itself only at the  $\tilde{R}_{2323} = \mathcal{T}_{2323}$  stage.

When  $K$  is equal to 1, the scaling multiple  $\lambda$  vanishes and the dilation factor  $\epsilon$  is 1. The Randers metric in question then reduces to the standard Riemannian one that  $S^3$  inherits from Euclidean  $\mathbb{R}^4$ . This case is not of interest to us because we want non-Riemannian Randers spaces. So let us impose the restriction

$$K > 1.$$

## 5. TWO EXPLICIT DESCRIPTIONS OF THE RESULTING RANDERS METRIC

Let us recapitulate by giving explicit formulas for the Finsler functions of the Randers spaces we have just obtained.

The first description is in terms of the non-holonomic frame  $\{e_1, e_2, e_3\}$  and its coframe  $\{\omega^1, \omega^2, \omega^3\}$ , both globally defined on the manifold  $M = S^3$ . The frame is non-holonomic because each  $e_p$  is a constant multiple of  $E_p$ , and the latter have nonzero Lie brackets amongst themselves (see §3).

Anyway, the Riemannian metric is

$$\tilde{a} = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3,$$

and the drift 1-form is

$$\tilde{b} = \pm \sqrt{\frac{K-1}{K}} \omega^1.$$

Expanding arbitrary tangent vectors as

$$y = y^1 e_1 + y^2 e_2 + y^3 e_3,$$

we see that the Finsler function is given by

$$F(x, y) = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2} \pm \sqrt{\frac{K-1}{K}} y^1.$$

It is remarkable that there is only an *implicit* dependence on the position  $x$ . We will revisit this deceptively simple formula in §7.

The second description of our Randers metric is in terms of natural coordinates. For this purpose, we used the following parametrization of  $S^3$ :

$$(x^o, x^1, x^2, x^3) = \frac{1}{\sqrt{1+x^2+y^2+z^2}} (c, x, y, z),$$

where  $c$  has the value  $+1$  when dealing with the *right* hemisphere, and the value  $-1$  when dealing with the *left* hemisphere. Obviously, the  $y$  here no longer denotes generic tangent vectors on  $S^3$ . Instead, it is part of the collection  $x, y, z$  that one typically uses when discussing Cartesian coordinates in Euclidean  $\mathbb{R}^3$ .

This parametrization has the advantage that it imposes no restriction on  $x, y, z$ , and can be visualized as follows. Consider for example the right hemisphere of  $S^3$ , centered at the origin  $O := (0, 0, 0, 0)$  of  $\mathbb{R}^4$ . Place a hyperplane  $\mathbb{R}^3$  tangent to the sphere at the East Pole  $(1, 0, 0, 0)$ . Points on this tangent hyperplane are of the form  $(1, x, y, z)$ . Now we assign coordinates to any given position  $P := (x^o, x^1, x^2, x^3)$  on the right hemisphere of  $S^3$ . Draw the straight line segment from the origin  $O$  to  $P$  and prolongate that until it intersects the tangent hyperplane. The point of intersection, being on that hyperplane, will have the form  $(1, x, y, z)$ . Just declare the natural coordinates of our point  $P$  to be  $(x, y, z)$ . Note that points along the equator of the right hemisphere correspond to points at infinity on the tangent hyperplane. A similar story holds for the left hemisphere.

The Riemannian metric  $\tilde{a}$  and the drift 1-form  $\tilde{b}$  are

$$\begin{aligned} \tilde{a} &= K \Theta^1 \otimes \Theta^1 + \Theta^2 \otimes \Theta^2 + \Theta^3 \otimes \Theta^3, \\ \tilde{b} &= \pm \sqrt{K-1} \Theta^1. \end{aligned}$$

Simple computations give:

$$\begin{aligned} \Theta^1 &= \frac{+cdx - zdy + ydz}{1+x^2+y^2+z^2}, \\ \Theta^2 &= \frac{+zdx + cdy - xdz}{1+x^2+y^2+z^2}, \\ \Theta^3 &= \frac{-ydx + xdy + cdz}{1+x^2+y^2+z^2}. \end{aligned}$$

Denote generic tangent vectors on  $S^3$  as

$$u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} .$$

Then the Finsler function for our Randers space is

$$F(x, y, z; u, v, w) = \alpha(x, y, z; u, v, w) + \beta(x, y, z; u, v, w),$$

with

$$\alpha = \frac{\sqrt{K(cu - zv + yw)^2 + (zu + cv - xw)^2 + (-yu + xv + cw)^2}}{1 + x^2 + y^2 + z^2} ,$$

$$\beta = \frac{\pm \sqrt{K-1} (cu - zv + yw)}{1 + x^2 + y^2 + z^2} .$$

As a reminder,  $c = +1$  for the right hemisphere, and  $c = -1$  for the left hemisphere. Some Maple programming will be carried out on this formula in the Appendix.

The Riemannian norm of the drift 1-form is

$$\|\tilde{b}\| = \sqrt{\frac{K-1}{K}} = \sqrt{1 - \frac{1}{K}} .$$

It is given by this value whether one is using the  $\tilde{a}$ -orthonormal frame or natural coordinates. Since  $K \geq 1$  (note:  $K = 1$  corresponds to the standard sphere  $S^3$ ), we see that  $\|\tilde{b}\|$  is strictly less than 1. This is both necessary and sufficient ([AIM], [BCS]) for  $F$  to be positive and strongly convex on the entire slit tangent bundle  $TM \setminus 0$ .

## 6. SPRAY CURVATURES AND FLAG CURVATURES

The flag curvature is defined much like the sectional curvature of Riemannian geometry. One begins with a connection, differentiates appropriately, and then performs some standard contractions. Let us first give a synopsis of the procedure.

The connection in use (from a family that contains the handful of name brand ones) gives rise to a  $hh$ -curvature tensor  $R_j^i{}_{kl}$ , which in the Riemannian case is precisely the familiar Riemann curvature. The precise formula of  $R_j^i{}_{kl}$  in terms of, say, the Chern connection, has been given a pedagogical treatment in [BCS]. Since that specific formula does not concern us in this paper, we shall omit it.

At any point  $x$  on  $M$ , one associates with each flagpole—a nonzero vector  $y$  in  $T_x M$ —the *spray* curvature

$$K^i{}_k := y^j R_j^i{}_{kl} y^l .$$

For those familiar with [BCS], this  $K^i_k$  is  $F^2$  times something called the ‘predecessor of the flag curvature.’ It is actually the first among two spray curvatures ([B3], [D]). The second one, using the version described in [D] and multiplying by  $F$ , is equal to the  $hv$ -curvature  ${}^bP_j^i{}_{kl}$  of the Berwald connection, but is irrelevant to the purpose of this paper.

Our notation  $K^i_k$  is the same as Berwald’s [B1]. Rund [R] uses  $H^i_k$  instead. The spray curvature is robust enough that it does not depend on our choice of connection. However, the above definition is often not an efficient way to compute the spray curvature because one has to obtain the tensor  $R_j^i{}_{kl}$  first.

A word of caution about notation. For us,  $K_{ik}$  shall simply mean  $g_{ij}K^j_k$ . It can be shown (see [BCS]) that  $K_{ik}$  is symmetric in  $i, k$ , and  $y^i K_{ik}, K_{ik}y^k$  both vanish. Rund did *not* subscribe to this system. He defined his  $H_{ik}$  by first tracing the indices  $j$  and  $l$  on  $\frac{1}{3}[(H^j_l)_{y^k} - (H^j_k)y^l]$ , forming an intermediate  $H_k$ , and then taking the  $y$ -partial  $(H_k)_{y^i}$ . See pages 129 and then 125 of [R]. As a result, his  $H_{ik}$  was not symmetric and  $y^i H_{ik}$  did not vanish for him.

The flagpole  $y$ , together with any vector  $V \in T_x M$  transversal to  $y$ , specifies a flag based at the point  $x \in M$ . The *flag* curvature of the resulting flag is the quantity

$$K(y, V) := \frac{V^i K_{ik} V^k}{g(y, y)g(V, V) - [g(y, V)]^2},$$

where  $g := g_{ij}dx^i \otimes dx^j$  is the fundamental tensor defined in §2. The Finsler metric is said to have constant flag curvature if  $K(y, V)$  has the same constant value  $K$  for any choice of  $y$  and  $V$ .

Specialize now to the case of constant flag curvature  $K$ . Euler’s theorem for homogeneous functions implies that  $g(y, y) = F^2(x, y)$ . Let’s substitute this into the above formula and rearrange it into the form

$$V^i K_{ik} V^k = K F^2 \{g(V, V) - [g(\ell, V)]^2\},$$

where  $\ell$  means  $y/F$ . As such, this statement makes formal sense even if  $V$  is not transversal to  $y$ . Since both sides are given by symmetric bilinear forms, a standard polarization identity gives

$$U^i K_{ik} W^k = K F^2 \{g(U, W) - g(\ell, U)g(\ell, W)\}.$$

In particular,

$$K_{ik} = K F^2 (g_{ik} - F_{y^i} F_{y^k}).$$

Here, we have used the identity  $g_{sj}\ell^s = F_{y^j}$ , again a consequence of Euler’s theorem. Raising the index  $i$  with the inverse matrix  $g^{ij}$  of the fundamental

tensor, we get

$$K^i_k = K F^2 \left( \delta^i_k - \frac{y^i}{F} F_{y^k} \right).$$

We shall use this criterion as the characterization of constant flag curvature.

Next, we describe a computationally friendlier way to access the spray curvature  $K^i_k$ . The history behind this better approach goes all the way back to Berwald [B1] in his study of path spaces and projective geometry. See Rund [R] for a careful exposition, and references therein.

One begins with the geodesic spray coefficients

$$G^i := \frac{1}{2} \gamma^i_{jk} y^j y^k,$$

where

$$\gamma^i_{jk} := g^{il} \frac{1}{2} \left( \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} + \frac{\partial g_{kl}}{\partial x^j} \right)$$

are the fundamental tensor's formal Christoffel symbols of the second kind. Our definition of  $G^i$  here is  $\frac{1}{2}$  times that given in the book [BCS]. As a result, the present definition agrees exactly with the one used in [AIM] and [R]. Chapter IV formula (6.3) in [R] tells us that

$$K^i_k = 2(G^i)_{x^k} - y^j (G^i)_{x^j y^k} - (G^i)_{y^j} (G^j)_{y^k} + 2G^j (G^i)_{y^j y^k}.$$

For ease of exposition, let us refer to this as Berwald's formula. It is an instructive exercise to manipulate the original expression for the spray curvature, namely  $y^j R^i_{jkl} y^l$ , into the one we just displayed. See §12.5B in [BCS] if guidance is needed.

The above formula expresses the spray curvature in terms of partial derivatives of the geodesic spray coefficients. As such, it is quite useful for machine computations of the spray curvature. The Maple codes that implement this sort of computation are given in the Appendix.

The said formula is also invaluable for calculating spray curvatures by hand. However, its efficient implementation for this purpose often requires some additional maneuvers as described in [S1]. To get us set up for §7, we specialize this technique to the Randers setting and present it here.

The geodesic spray coefficients of every Randers metric is equal to that of the underlying Riemannian metric, plus a perturbation term. Symbolically, we write

$$G^i = \tilde{G}^i + \zeta^i.$$

Here,

$$\tilde{G}^i := \frac{1}{2} \tilde{\gamma}^i_{jk} y^j y^k,$$

where

$$\tilde{\gamma}^i_{jk} := \tilde{a}^{il} \frac{1}{2} \left( \frac{\partial \tilde{a}_{lj}}{\partial x^k} - \frac{\partial \tilde{a}_{jk}}{\partial x^l} + \frac{\partial \tilde{a}_{kl}}{\partial x^j} \right)$$

are the Christoffel symbols of the second kind of the Riemannian metric  $\tilde{a}$ . The specific form of the perturbation term  $\zeta^i$  has been worked out in [BCS], but shall not concern us until §7.

Substitute the above  $G^i$  into Berwald's formula for the spray curvature. After covariantizing (to be explained below) all  $x$ -partial derivatives of  $\zeta^i$  and using Euler's theorem for homogeneous functions whenever appropriate, we get

$$K^i_k = \tilde{K}^i_k + \left\{ 2\zeta^i_{|k} - y^j(\zeta^i_{|j})_{y^k} - (\zeta^i)_{y^j}(\zeta^j)_{y^k} + 2\zeta^j(\zeta^i)_{y^j y^k} \right\}.$$

This formula will be our centerpiece in §7. The first term on the right-hand side,  $\tilde{K}^i_k$ , is the spray curvature of the Riemannian metric  $\tilde{a}$ . It can readily be calculated in the  $S^3$  examples because the Riemann curvature  $\tilde{R}_{jikl}$  of  $\tilde{a}$  is already at our disposal (§4).

The remaining group of terms on the right-hand side involves horizontal covariant differentiation on the slit tangent bundle  $TM \setminus 0$ . Let us explain how those are carried out. The geodesic spray coefficients  $\tilde{G}^i$  are first used to define

$$\tilde{N}^i_j := (\tilde{G}^i)_{y^j}$$

and then

$$\frac{\delta}{\delta x^j} := \frac{\partial}{\partial x^j} - \tilde{N}^l_j \frac{\partial}{\partial y^l}.$$

The vector fields  $\frac{\delta}{\delta x^j}$  are declared to be horizontal—the horizontal lift of  $\frac{\partial}{\partial x^j}$  to  $TM \setminus 0$ . And the  $\tilde{N}^i_j$  are said to have produced a nonlinear Ehresmann connection on the slit tangent bundle. In order to achieve the sought horizontal covariant differentiation on  $\zeta^i$ , we horizontally lift the Levi-Civita (Christoffel) connection of  $\tilde{a}$ , and then let the resulting object act on  $\zeta^i$ . This may seem abstract but is operationally quite simple:

$$\zeta^i_{|j} := \frac{\delta}{\delta x^j} \zeta^i + \zeta^l \tilde{\gamma}^i_{lj}.$$

Incidentally, note that

$$\tilde{\gamma}^i_{jk} = (\tilde{G}^i)_{y^j y^k}.$$

This complements nicely with the fact that the nonlinear connection is given by the first  $y$ -partial derivatives of  $\tilde{G}^i$ . Thus

$$(\zeta^i)_{x^j} = \zeta^i_{|j} + (\zeta^i)_{y^l}(\tilde{G}^l)_{y^j} - \zeta^l(\tilde{G}^i)_{y^l y^j}.$$

The use of this formula is what we mean above by covariantizing the  $x$ -partial derivatives of  $\zeta^i$ . We did that to make each term inside the expression  $\{\dots\}$  manifestly tensorial. Of course, the quantity  $\{\dots\}$  as a whole is already tensorial, whether we covariantize or not. There is no need to covariantize the  $y$ -partial derivatives because they already transform tensorially (see [BCS] for an exposition).

## 7. DIRECT PROOF OF CONSTANT POSITIVE FLAG CURVATURE

As we mentioned in the Introduction, the two published proofs ([YS] and [M]) of the Yasuda–Shimada theorem both contain arguments that we find difficult to follow. Given that, it is prudent to prove directly that our Randers metrics have constant positive flag curvature  $K$ . The numerical evidence documented in the Appendix helps sustain such a bare-hands’ proof. Strategically, we compute both sides of the constant curvature criterion

$$K^p_r = K F^2 \left( \delta^p_r - \frac{y^p}{F} F_{y^r} \right)$$

and ascertain that they are indeed equal. The calculations rely on Berwald’s formula, and will be done with the help of an  $\tilde{a}$ -orthonormal frame instead of natural coordinates. The resulting proof is geometrical and is distinctly different from the approaches used in [YS], [M]. It presents a new perspective on the subject, and is worthy of further development.

### 7.1. The right-hand side of the constant curvature criterion.

Let us begin with the formula for  $F$  in terms of the  $\tilde{a}$ -orthonormal frame  $\{e_1, e_2, e_3\}$  on  $S^3$ . This was first given in §5. For ease of presentation, we relabel the tangent space coordinates  $y^1, y^2, y^3$  as  $u, v, w$ , respectively. These coordinates arise from the expansion of arbitrary tangent vectors  $y$  in terms of the basis  $\{e_1, e_2, e_3\}$ . The cosmetically altered formula for  $F$  now reads

$$F(x, y) = \sqrt{u^2 + v^2 + w^2} + \mathfrak{s} \sqrt{\frac{K-1}{K}} u =: \alpha + \beta .$$

Here,  $\mathfrak{s} = \pm 1$  keeps track of the two choices of the drift 1-form. See the end of §4. These choices of sign have nothing to do with the right or left hemisphere of  $S^3$ . The latter are associated with the parameter  $c$  of §5.

Almost by inspection, we have

$$\begin{aligned} F_u &= \frac{u}{\alpha} + \mathfrak{s} \sqrt{\frac{K-1}{K}} , \\ F_v &= \frac{v}{\alpha} , \\ F_w &= \frac{w}{\alpha} . \end{aligned}$$

Using these, the right-hand side

$$K \tau^p_r := K F^2 \left( \delta^p_r - \frac{y^p}{F} F_{y^r} \right)$$

of our constant curvature criterion is readily computed. We find that

$$\begin{pmatrix} K\tau_1^1 & K\tau_2^1 & K\tau_3^1 \\ K\tau_1^2 & K\tau_2^2 & K\tau_3^2 \\ K\tau_1^3 & K\tau_2^3 & K\tau_3^3 \end{pmatrix}$$

is equal to

$$\begin{pmatrix} \frac{F}{\alpha}K(v^2+w^2) & -\frac{F}{\alpha}Kuv & -\frac{F}{\alpha}Kuw \\ -\frac{F}{\alpha}Kuv - F Kv \mathfrak{s} \sqrt{\frac{K-1}{K}} & FK(F - \frac{v^2}{\alpha}) & -\frac{F}{\alpha}Kvw \\ -\frac{F}{\alpha}Kuw - F Kw \mathfrak{s} \sqrt{\frac{K-1}{K}} & -\frac{F}{\alpha}Kvw & FK(F - \frac{w^2}{\alpha}) \end{pmatrix}.$$

## 7.2. Perturbation terms in the geodesic spray coefficients.

The left-hand side of our constant curvature criterion is the spray curvature  $K^p_r$ . These nine components can be calculated using Berwald's formula. To this end, we start with the geodesic spray coefficients  $G^i$ . As we mentioned in §6, the ones used in this paper, *unlike* those of [BCS], already have the factor of  $\frac{1}{2}$  built in. It is shown in the same reference that for Randers metrics,

$$G^i = \tilde{G}^i + \zeta^i,$$

where  $\tilde{G}^i := \frac{1}{2} \tilde{\gamma}^i_{jk} y^j y^k$  are the geodesic spray coefficients of the underlying Riemannian metric  $\tilde{a}$ .

The perturbation terms  $\zeta^i$  transform like the components of a tensor. They do so because

$$\zeta^i = \frac{1}{2} (\gamma^i_{jk} - \tilde{\gamma}^i_{jk}) y^j y^k,$$

showing that  $\zeta$  arises from the difference of two connections. More abstractly, consider the vector bundle  $TM$  that sits over  $M$ . Using the projection map  $\pi : TM \setminus 0 \rightarrow M$ , we can pull that back to obtain a vector bundle  $\pi^*TM$  that sits over the manifold  $TM \setminus 0$ . This just means that over each point  $(x, y) \in TM \setminus 0$ , we have erected a copy of  $T_x M$ . Our  $\zeta^i$  then transforms like a section of this pulled-back vector bundle.

The coordinate bases  $\{\partial_{x^i}\}$ , the Riemannian metric  $\tilde{a}$ , together with the field of  $\tilde{a}$ -orthonormal frames  $\{e_p\}$ , can all be transplanted to the fibres of  $\pi^*TM$  (note: *not* to  $TM \setminus 0$ ). As on  $M$ , the transplants of  $\{\partial_{x^i}\}$  and  $\{e_p\}$  are related through the 3-bein  $u_p^i$  and its matrix inverse  $v^p_i$ :

$$e_p = u_p^i \partial_{x^i}, \quad \omega^p = v^p_i dx^i.$$

Each  $u_p^i$  and  $v^p_i$  is a function on  $M$ . In other words, they depend on  $x$  only.

The tensorial property mentioned above allows us to transform the coordinate description  $\zeta^i$  into the orthonormal frame description

$$\zeta^p := v^p_i \zeta^i.$$

With this in mind, and the factor of  $\frac{1}{2}$  emphasized earlier, an expression derived in [BCS] is transcribed. It gives

$$\zeta^p := \frac{1}{2} \alpha \tilde{b}_{q|r} \left\{ (\tilde{a}^{pq} y^r - \tilde{a}^{pr} y^q) + \ell^p (y^q \tilde{b}^r - y^r \tilde{b}^q) + \frac{\ell^p y^q y^r}{\alpha} \right\}.$$

Here,  $\ell^p := y^p/F$  is to be distinguished from  $\tilde{\ell}^p := y^p/\alpha$ . We shall show that these  $\zeta^p$  components are actually quite simple.

In §4, we determined the intermediate constants  $\epsilon$  and  $\lambda$ . These can be used to update the Levi-Civita (Christoffel) connection  $\omega_q^p$  of the Riemannian metric  $\tilde{a}$ , the components of the 1-form  $\tilde{b}$ , and its covariant derivatives  $\tilde{b}_{q|r}$ . A quick glance at §3 and §4 tells us that the updated versions read:

$$\begin{pmatrix} \omega_1^1 & \omega_1^2 & \omega_1^3 \\ \omega_2^1 & \omega_2^2 & \omega_2^3 \\ \omega_3^1 & \omega_3^2 & \omega_3^3 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{K}\omega^3 & \sqrt{K}\omega^2 \\ \sqrt{K}\omega^3 & 0 & \left(\sqrt{K} - \frac{2}{\sqrt{K}}\right)\omega^1 \\ -\sqrt{K}\omega^2 & \left(\frac{2}{\sqrt{K}} - \sqrt{K}\right)\omega^1 & 0 \end{pmatrix},$$

together with

$$\tilde{b}_1 = \mathfrak{s} \sqrt{\frac{K-1}{K}}, \quad \tilde{b}_2 = 0, \quad \tilde{b}_3 = 0,$$

and

$$\begin{pmatrix} \tilde{b}_{1|1} & \tilde{b}_{1|2} & \tilde{b}_{1|3} \\ \tilde{b}_{2|1} & \tilde{b}_{2|2} & \tilde{b}_{2|3} \\ \tilde{b}_{3|1} & \tilde{b}_{3|2} & \tilde{b}_{3|3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathfrak{s}\sqrt{K-1} \\ 0 & +\mathfrak{s}\sqrt{K-1} & 0 \end{pmatrix}.$$

Since the present  $\tilde{b}_{q|r}$  is skew-symmetric and most of its components are zero, the perturbation term  $\zeta^p$  simplifies drastically to:

$$\zeta^p = \alpha \mathfrak{s} \sqrt{K-1} \left\{ \tilde{a}^{p3} y^2 - \tilde{a}^{p2} y^3 + \ell^p (y^3 \tilde{b}^2 - y^2 \tilde{b}^3) \right\}.$$

Furthermore, we are in an  $\tilde{a}$ -orthonormal frame, so  $\tilde{b}^r := \tilde{a}^{rs} \tilde{b}_s = \delta^{rs} \tilde{b}_s$  is numerically the same as  $\tilde{b}_r$ . Given that  $\tilde{b}_2$  and  $\tilde{b}_3$  are both zero, the terms multiplying  $\ell^p$  drop out, and

$$\zeta^p = \alpha \mathfrak{s} \sqrt{K-1} \left\{ \delta^{p3} y^2 - \delta^{p2} y^3 \right\}.$$

In other words,

$$\begin{aligned} \zeta^1 &= 0, \\ \zeta^2 &= -\mathfrak{s} \sqrt{K-1} \mathbf{w} \alpha, \\ \zeta^3 &= +\mathfrak{s} \sqrt{K-1} \mathbf{v} \alpha. \end{aligned}$$

Recall that in the  $\tilde{a}$ -orthonormal frame description, the quantity  $\alpha$  is simply  $\sqrt{\mathbf{u}^2 + \mathbf{v}^2 + \mathbf{w}^2}$ . So the components  $\zeta^p$  contain no explicit dependence on

$x$ . However,  $u, v, w$  are  $y^p$ , which can be expressed as  $v^p_i y^i$ . Thus the dependence on  $x$  is there, albeit implicitly through  $v^p_i$ .

### 7.3. The spray curvature of the Riemannian metric.

In §6, we described a refined understanding of Berwald's formula. It is a case study of the general strategy discussed in [S1]. The essence is that given the decomposition  $G^i = \tilde{G}^i + \zeta^i$ , the spray curvature splits as well:

$$K^i_k = \tilde{K}^i_k + \left\{ 2\zeta^i_{|k} - y^j(\zeta^i_{|j})_{y^k} - (\zeta^i)_{y^j}(\zeta^j)_{y^k} + 2\zeta^j(\zeta^i)_{y^j y^k} \right\}.$$

Each side of this equation transforms like a section of the pulled-back tensor bundle  $\pi^*TM \otimes \pi^*T^*M$ , which sits over  $TM \setminus 0$ .

Contract both sides of the above equation with  $v^p_i u^k_r$ . This converts its expression from the coordinate to the  $\tilde{a}$ -orthonormal perspective:

$$K^p_r = \tilde{K}^p_r + \left\{ 2\zeta^p_{|r} - y^q(\zeta^p_{|q})_{y^r} - (\zeta^p)_{y^q}(\zeta^q)_{y^r} + 2\zeta^q(\zeta^p)_{y^q y^r} \right\}.$$

The first term on the right-hand side is the spray curvature of the Riemannian metric  $\tilde{a}$ . Since the Riemann curvature tensor  $\tilde{R}_{qprs}$  of  $\tilde{a}$  has been determined in §4, we can use that information to compute the said spray curvature:

$$\tilde{K}^p_r = y^q \tilde{R}^p_{qrs} y^s.$$

Keep in mind that we are in an  $\tilde{a}$ -orthonormal frame, so the numerical values of tensor components are unaffected by raising or lowering any index, as long as it is done using  $\tilde{a}$ . Fairly routine calculations tell us that the nine spray curvatures

$$\begin{pmatrix} \tilde{K}^1_1 & \tilde{K}^1_2 & \tilde{K}^1_3 \\ \tilde{K}^2_1 & \tilde{K}^2_2 & \tilde{K}^2_3 \\ \tilde{K}^3_1 & \tilde{K}^3_2 & \tilde{K}^3_3 \end{pmatrix}$$

are equal to

$$\begin{pmatrix} K(v^2 + w^2) & -Kuv & -Kuw \\ -Kuv & Ku^2 + (4 - 3K)w^2 & (3K - 4)vw \\ -Kuw & (3K - 4)vw & Ku^2 + (4 - 3K)v^2 \end{pmatrix}.$$

The symmetry here is expected because  $\tilde{K}^p_r := \tilde{a}^{ps} \tilde{K}_{sr}$  has the same numerical value as  $\tilde{K}_{pr}$ , and the latter is symmetric by general principles. In contradistinction to this, the indices on  $K^p_r$  are raised and lowered with  $g$  rather than  $\tilde{a}$ . Since the basis we are using on the fibres of  $\pi^*TM$  is not  $g$ -orthonormal, we do *not* expect  $K^p_r$  to be numerically equal to the symmetric  $K_{pr}$ .

#### 7.4. A simplified formula for horizontal covariant derivatives.

In the refined version of Berwald's formula, we have the terms

$$2\zeta^p_{|r} - y^q(\zeta^p_{|q})_{y^r} - (\zeta^p)_{y^q}(\zeta^q)_{y^r} + 2\zeta^q(\zeta^p)_{y^q y^r}.$$

These involve both the  $y$ -partial derivatives of  $\zeta^p$  as well as its horizontal covariant derivatives. The purpose of this subsection is to derive a simplified formula for these horizontal covariant derivatives, one that will facilitate their computation in §7.5.

In §6, we explained how the horizontal covariant derivatives are to be carried out in the context of natural coordinates. Namely,

$$\zeta^i_{|k} = \frac{\delta}{\delta x^k} \zeta^i + \zeta^l \tilde{\gamma}^i_{lk}.$$

Note that  $\tilde{\gamma}^i_{lj} dx^j$  are the connection forms with respect to the coordinate basis. When these connection forms are pulled back to  $TM \setminus 0$  and evaluated on the horizontal lifts (of  $\partial_{x^k}$  to  $TM \setminus 0$ )

$$\frac{\delta}{\delta x^k} := \frac{\partial}{\partial x^k} - \tilde{N}^i_k \frac{\partial}{\partial y^i},$$

they see only the  $\partial_{x^k}$  part but not the  $\partial_{y^i}$  part. The above expression for  $\zeta^i_{|k}$  can then be recast as

$$\zeta^i_{|k} = \{d\zeta^i + \zeta^l (\tilde{\gamma}^i_{lj} dx^j)\} \left( \frac{\delta}{\delta x^k} \right).$$

In the current subsection, we are using the basis  $\{e_p \otimes \omega^r\}$  (instead of the coordinate one) on the pulled-back bundle  $\pi^*TM \otimes \pi^*T^*M$ . The analogous formula for  $\zeta^p_{|r}$  is

$$\zeta^p_{|r} = \{d\zeta^p + \zeta^s \omega_s^p\}(\hat{e}_r),$$

where  $\omega_s^p$  are the connection forms displayed in §7.2. The vector field  $\hat{e}_r$  is the horizontal lift of  $e_r$ , and has the formula

$$\hat{e}_r := e_r - y^s \omega_s^p(e_r) \partial_{y^p}.$$

This is not unreasonable, given that in the horizontal lift of  $\partial_{x^k}$ , the non-linear connection can be re-expressed as follows:

$$\tilde{N}^i_k = \tilde{\gamma}^i_{lk} y^l = y^l (\tilde{\gamma}^i_{lj} dx^j)(\partial_{x^k}).$$

Note, however, that we have yet to define  $e_r$ , and for that matter  $\partial_{x^k}$ , as objects on  $TM \setminus 0$ . Here are the definitions:

- The 1-forms  $dx^k$  are pulled-back to  $TM \setminus 0$  and given the same name. They are then complemented by the  $dy^k$  (with  $y^k$  coming from  $y^k \partial_{x^k}$ ) to form a coordinate basis for the cotangent bundle of  $TM \setminus 0$ . The natural dual of this basis is denoted  $\{\partial_{x^k}, \partial_{y^k}\}$ . The objects in this basis are local vector fields on  $TM \setminus 0$ . They are the ones that enter

the definition of  $\frac{\delta}{\delta x^k}$ . It is simply an abuse of notation to employ the same symbol  $\partial_{x^k}$  for the vector fields here and those on  $M$ .

- In the same spirit, we pull back the 1-forms  $\omega^r$  to  $TM \setminus 0$  and retain the same name. These are complemented by the  $dy^r$  (with  $y^r$  arising from  $y^r e_r$ ) to give a non-holonomic basis. Denote the natural dual of this basis as  $\{e_r, \partial_{y^r}\}$ . Just like the  $\partial_{x^k}$  case, we are abusing the notation when we call these  $e_r$  and those on  $M$  by the same name. Nonetheless, we now know what  $e_r$  means on  $TM \setminus 0$ , and the formula for the horizontal lift  $\hat{e}_r$  makes good sense.

Do not confuse the  $e_r$  and  $\partial_{x^k}$  we just defined with the transplants (see §7.2) that live on the pulled-back vector bundle  $\pi^*TM$ , even though they share the same notation merely for the sake of economy.

One can't help but wonder how  $e_r$  is related to  $\partial_{x^k}$  as vector fields on  $TM \setminus 0$ . To find out, take an arbitrary differentiable function  $f$  on  $TM \setminus 0$  and let  $e_r$  act on it. By the chain rule,

$$(df)(e_r) = \{(\partial_{x^k} f) dx^k + (\partial_{y^k} f) dy^k\} (e_r).$$

- Note that  $y^k = y^s u_s^k$  gives  $dy^k = u_s^k dy^s + y^s du_s^k$ . It is to minimize confusion in this statement that we have chosen the font  $y^s$  throughout the paper, when dealing with orthonormal expansions of  $y$ .
- The relation  $dx^k = u_q^k \omega^q$  is valid on  $TM \setminus 0$  because it holds on  $M$  and the pull-back operation preserves algebraic statements.
- Since  $\{e_p, \partial_{y^p}\}$  is by definition the natural dual of  $\{\omega^p, dy^p\}$ , one must have  $\omega^q(e_r) = \delta_r^q$  and  $dy^s(e_r) = 0$ .

Substituting these observations into our chain rule statement, and removing the test function  $f$  afterwards, we obtain the somewhat surprising formula

$$e_r = u_r^k \partial_{x^k} + y^s du_s^k (e_r) \partial_{y^k} \quad \text{on } TM \setminus 0.$$

In contradistinction to that, a similar calculation gives

$$\partial_{y^r} = u_r^k \partial_{y^k} \quad \text{on } TM \setminus 0.$$

Here, one does have to invoke  $(du_s^k)(\partial_{y^r}) = 0$ . This holds because  $u_s^k$  is a function on  $M$ , so its differential is a linear combination of  $dx^j$  and hence of  $\omega^q$ . Such a relation is preserved under pull-back, and  $\omega^q(\partial_{y^r}) = 0$ .

Let us return to the horizontal covariant derivative

$$\zeta^p_{|r} = \{d\zeta^p + \zeta^s \omega_s^p\} (\hat{e}_r),$$

with  $\hat{e}_r := e_r - y^s \omega_s^t (e_r) \partial_{y^t}$ . This can stand two more reductions before we put it to use in §7.5.

- On  $M$ , the connection forms  $\omega_s^p$  are linear combinations of the  $\omega^q$ , which are in turn linear combinations of the  $dx^i$ . This remains so

under the pull-back to  $TM \setminus 0$ , so  $\omega_s^p(\partial_{y^k}) = 0$ . Since  $\partial_{y^t} = u_t^k \partial_{y^k}$ , we must have  $\omega_s^p(\partial_{y^t}) = 0$  as well. Therefore

$$\omega_s^p(\hat{e}_r) = \omega_s^p(e_r).$$

- At the end of §7.2, we deduced that the components  $\zeta^p$  depend only on  $u, v, w$ , namely  $y^q$ . Formally taking the differential yields  $d\zeta^p = (\zeta^p)_{y^q} dy^q$ . On the other hand, linear algebra says that  $d\zeta^p = (d\zeta^p)(e_q) \omega^q + (d\zeta^p)(\partial_{y^q}) dy^q$ . Comparing the two statements gives  $(d\zeta^p)(e_q) = 0$ , which in turn allows us to conclude that

$$(d\zeta^p)(\hat{e}_r) = -y^s \omega_s^t(e_r) (\zeta^p)_{y^t}.$$

The two reductions above lead to the following:

$$\zeta^p_{|r} = \{\zeta^s \omega_s^p - (\zeta^p)_{y^t} y^s \omega_s^t\} (e_r).$$

We shall use this formula in §7.5 to compute horizontal covariant derivatives. By the way, in the context of [BCS], the  $\omega_q^p$  here happens to be the Chern connection forms of the Riemannian metric  $\tilde{a}$ .

### 7.5. Derivatives of the perturbation terms.

In this subsection, we tabulate the requisite  $y^r$ -partial derivatives of  $\zeta^p$ . Then we use the concluding formula of §7.4, together with  $\omega_q^p$  from §7.2 and  $\omega^s(e_r) = \delta^s_r$ , to calculate the horizontal covariant derivatives that we need. Recall that

$$\zeta^1 = 0, \quad \zeta^2 = -\mathfrak{s}\sqrt{K-1} w \alpha, \quad \zeta^3 = +\mathfrak{s}\sqrt{K-1} v \alpha,$$

with

$$\alpha := \sqrt{u^2 + v^2 + w^2}.$$

Here,  $u = y^1, v = y^2, w = y^3$  and, from §7.1,  $\mathfrak{s} = \pm 1$ . Thus all  $y^r$ -partials of  $\zeta^1$  are zero. On the other hand,  $\zeta^1_{|r}$  do not necessarily vanish because  $\zeta^2$  and  $\zeta^3$  are involved in the calculations.

Let us list the results. We introduce some abbreviations for ubiquitous quantities in order to reduce clutter.

$$\text{With } \diamond := \frac{\mathfrak{s}}{\alpha} \sqrt{K-1} :$$

$$\begin{aligned} (\zeta^2)_{y^1} &= -\diamond (uw) & (\zeta^3)_{y^1} &= +\diamond (uv) \\ (\zeta^2)_{y^2} &= -\diamond (vw) & (\zeta^3)_{y^2} &= +\diamond (u^2 + 2v^2 + w^2) \\ (\zeta^2)_{y^3} &= -\diamond (u^2 + v^2 + 2w^2) & (\zeta^3)_{y^3} &= +\diamond (vw) \end{aligned}$$

$$\text{With } \star := \frac{\mathfrak{s}}{\alpha^3} \sqrt{K-1} :$$

$$\begin{array}{ll}
(\zeta^2)_{y^1 y^2} = +\star(uvw) & (\zeta^3)_{y^1 y^2} = +\star u(u^2 + w^2) \\
(\zeta^2)_{y^1 y^3} = -\star u(u^2 + v^2) & (\zeta^3)_{y^1 y^3} = -\star(uvw) \\
(\zeta^2)_{y^2 y^2} = -\star w(u^2 + w^2) & (\zeta^3)_{y^2 y^2} = +\star v(3u^2 + 2v^2 + 3w^2) \\
(\zeta^2)_{y^2 y^3} = -\star v(u^2 + v^2) & (\zeta^3)_{y^2 y^3} = +\star w(u^2 + w^2) \\
(\zeta^2)_{y^3 y^3} = -\star w(3u^2 + 3v^2 + 2w^2) & (\zeta^3)_{y^3 y^3} = +\star v(u^2 + v^2)
\end{array}$$

With  $\heartsuit := \alpha \mathfrak{s} \sqrt{K-1} \sqrt{K}$ :

$$\begin{array}{lll}
\zeta^1_{|1} = 0 & \zeta^2_{|1} = 0 & \zeta^3_{|1} = 0 \\
\zeta^1_{|2} = -\heartsuit v & \zeta^2_{|2} = +\heartsuit u & \zeta^3_{|2} = 0 \\
\zeta^1_{|3} = -\heartsuit w & \zeta^2_{|3} = 0 & \zeta^3_{|3} = +\heartsuit u
\end{array}$$

Lastly,

With  $\clubsuit := \frac{1}{\alpha} \mathfrak{s} \sqrt{K-1} \sqrt{K}$ :

$$\begin{array}{ll}
\left(\zeta^2_{|2}\right)_{y^1} = +\clubsuit(2u^2 + v^2 + w^2) & \left(\zeta^3_{|3}\right)_{y^1} = +\clubsuit(2u^2 + v^2 + w^2) \\
\left(\zeta^2_{|2}\right)_{y^2} = +\clubsuit(uv) & \left(\zeta^3_{|3}\right)_{y^2} = +\clubsuit(uv) \\
\left(\zeta^2_{|2}\right)_{y^3} = +\clubsuit(uw) & \left(\zeta^3_{|3}\right)_{y^3} = +\clubsuit(uw)
\end{array}$$

These two columns are equal because  $(\zeta^2_{|2}) = (\zeta^3_{|3})$ . Also, we have

$$\begin{array}{ll}
\left(\zeta^1_{|2}\right)_{y^1} = -\clubsuit(uv) & \left(\zeta^1_{|3}\right)_{y^1} = -\clubsuit(uw) \\
\left(\zeta^1_{|2}\right)_{y^2} = -\clubsuit(u^2 + 2v^2 + w^2) & \left(\zeta^1_{|3}\right)_{y^2} = -\clubsuit(vw) \\
\left(\zeta^1_{|2}\right)_{y^3} = -\clubsuit(vw) & \left(\zeta^1_{|3}\right)_{y^3} = -\clubsuit(u^2 + v^2 + 2w^2)
\end{array}$$

### 7.6. The spray curvature of our Randers metric.

According to §7.3, the spray curvature of the Randers metric

$$F := \alpha + \beta$$

has the structure

$$K^p_r = \tilde{K}^p_r + \mathcal{E}^p_r,$$

where

$$\mathcal{E}^p_r := 2\zeta^p_{|r} - y^q(\zeta^p_{|q})_{yr} - (\zeta^p)_{y^q}(\zeta^q)_{yr} + 2\zeta^q(\zeta^p)_{y^q y^r}.$$

The term  $\tilde{K}^p_r$  represents the nine spray curvatures of the underlying Riemannian metric  $\tilde{a}$ . We have already calculated those in §7.3.

Using the derivatives listed in §7.5, the above quantities  $\mathcal{E}_r^p$  are computed. We find that:

$$\begin{aligned}\mathcal{E}_1^1 &= +\clubsuit u(v^2 + w^2), \\ \mathcal{E}_2^1 &= -\clubsuit v(u^2), \\ \mathcal{E}_3^1 &= -\clubsuit w(u^2); \\ \mathcal{E}_1^2 &= -\clubsuit v(2u^2 + v^2 + w^2) - (K-1)uv, \\ \mathcal{E}_2^2 &= +\clubsuit u(2u^2 + v^2 + 2w^2) + (K-1)(u^2 + 4w^2), \\ \mathcal{E}_3^2 &= -\clubsuit uvw - 4(K-1)vw; \\ \mathcal{E}_1^3 &= -\clubsuit w(2u^2 + v^2 + w^2) - (K-1)uw, \\ \mathcal{E}_2^3 &= -\clubsuit uvw - 4(K-1)vw, \\ \mathcal{E}_3^3 &= +\clubsuit u(2u^2 + 2v^2 + w^2) + (K-1)(u^2 + 4v^2).\end{aligned}$$

Adding these  $\mathcal{E}_r^p$  to the corresponding  $\tilde{K}_r^p$  of §7.3, we obtain the spray curvatures  $K_r^p$ . After that, it is a matter of routine algebra to make sure each  $K_r^p$  is equal to the  $K\tau_r^p$  calculated in §7.1. Such is indeed the case. This means that the Randers metric in question satisfies the criterion

$$K_r^p = K F^2 \left( \delta_r^p - \frac{y^p}{F} F_{y^r} \right).$$

Therefore it has constant flag curvature  $K$  and our proof is complete.

## 8. DISCUSSION

Given any Finsler manifold  $(M, F)$ , its geodesics are paths in  $M$ , just like the Riemannian case. The defining equation for geodesics with constant Finslerian speed is

$$\ddot{x}^i + \gamma^i_{jk} \dot{x}^j \dot{x}^k = 0.$$

In terms of the coefficients  $G^i$  of §6, this system of second order quasi-linear equations can be rewritten as

$$\ddot{x}^i + 2G^i = 0.$$

This is one reason the  $G^i$  are called the geodesic spray coefficients.

A Finsler manifold is said to be projectively flat if  $M$  can be covered by privileged coordinate charts in which the above quasi-linear system of ordinary differential equations becomes a linear system. Projective flatness has been characterized by Douglas [D] in his study of path spaces. When specialized to the Finsler setting, his result states that:

A Finsler manifold  $(M, F)$  of dimension  $\geq 3$  is projectively flat if and only if its projective Weyl and Douglas tensors both vanish.

A slightly modified characterization, due to Berwald [B2], applies to dimension 2. There, the vanishing of the Weyl tensor is replaced by another criterion. See page 144 of Rund [R], or [S1], for a detailed account and further references.

The projective Weyl tensor is to be distinguished from the conformal Weyl tensor in Riemannian geometry. The latter is not defined in dimension 2 and always vanishes in dimension 3. It turns out that the vanishing of the projective Weyl tensor (which has four indices) is equivalent to the vanishing of its reduced version  $W^i_k$ . This is explained on pages 141 and 142 of [R].

The tensor  $W^i_k$  is defined in terms of the spray curvature and its derivatives. To that end, one first constructs the scalar quantity

$$\mathfrak{K} := \frac{1}{n-1} K^i_i.$$

Comparing with §7.6 of [BCS], we see that this scalar  $\mathfrak{K}$  can be interpreted as  $F^2$  times the average of  $n-1$  appropriately chosen flag curvatures. Using  $\mathfrak{K}$ , the reduced projective Weyl tensor is defined as

$$W^i_k := K^i_k - \mathfrak{K} \delta^i_k - \frac{1}{n+1} y^i \left\{ (K^j_k)_{y^j} - (\mathfrak{K})_{y^k} \right\}.$$

A result of Matsumoto's (see [AIM]) says that the vanishing of  $W^i_k$  is both necessary and sufficient for the flag curvature to depend at most on the position  $x$  and the flagpole  $y \in T_x M$ , but not on the transverse edges (denoted  $V$  in §6). In particular, since our Randers metrics on  $S^3$  have constant flag curvature  $K$ , we deduce that  $W^i_k$  must vanish for all these examples. This can also be verified by a direct computation.

The Douglas tensor  $D^i_{jkl}$  (denoted  $B^i_{jkl}$  in [R]) is defined as follows:

$$\begin{aligned} D^i_{jkl} := & (G^i)_{y^j y^k y^l} \\ & - \frac{1}{n+1} \left\{ \delta^i_j (G^h)_{y^h y^k y^l} + \delta^i_k (G^h)_{y^h y^l y^j} + \delta^i_l (G^h)_{y^h y^j y^k} \right\} \\ & - \frac{1}{n+1} y^i (G^h)_{y^h y^j y^k y^l}. \end{aligned}$$

In [BM], Básco and Matsumoto proved that a Randers space has vanishing Douglas tensor if and only if the drift 1-form  $\tilde{b}$  is closed. For our examples,  $\tilde{b}$  is a constant multiple of  $\Theta^1$ , see §4. Since  $d\Theta^1 = 2\Theta^2 \wedge \Theta^3 \neq 0$ , we conclude that the Douglas tensor does not vanish for our Randers metrics.

Now we know that each member among our family of Randers metrics on  $S^3$  has vanishing projective Weyl tensor but nonzero Douglas tensor. Therefore, by Douglas' theorem, none of them can be projectively flat.

## APPENDIX A. SOME NUMERICAL EVIDENCE

The purpose of this Appendix is to apply Maple to the Randers metric

$$F(x, y, z; u, v, w) = \alpha(x, y, z; u, v, w) + \beta(x, y, z; u, v, w)$$

in natural coordinates, with

$$\alpha = \frac{\sqrt{K(cu - zv + yw)^2 + (zu + cv - xw)^2 + (-yu + xv + cw)^2}}{1 + x^2 + y^2 + z^2},$$

$$\beta = \frac{\pm\sqrt{K-1} (cu - zv + yw)}{1 + x^2 + y^2 + z^2}.$$

Recall that  $c = +1$  for the right hemisphere of  $S^3$ , and  $c = -1$  for the left hemisphere.

In Finsler geometry, it is not uncommon for simple formulae to quickly mushroom into unmanageable expressions. We find that machine computations have consistently extracted useful information and insights, to the point that meaningful followup questions can be asked. We value every opportunity to cultivate this synergy between Finsler geometry and modern computing. It is hoped that by producing the Maple codes here, we are initiating other geometers into a fruitful aspect of experimental mathematics.

Here is our plan:

- We first use Maple to calculate the spray curvature *à la* Berwald:

$$K^i_k = 2(G^i)_{x^k} - y^j(G^i)_{x^j y^k} - (G^i)_{y^j}(G^j)_{y^k} + 2G^j(G^i)_{y^j y^k}.$$

- After that, we ask Maple to check whether  $F$  satisfies the characterization of having constant flag curvature  $K$ :

$$K^i_k = K F^2 \left( \delta^i_k - \frac{y^i}{F} F_{y^k} \right).$$

#### A.1. The Finsler function in natural coordinates.

```
> P:=c*u+y*w-z*v;
> Q:=c*v+z*u-x*w;
> R:=c*w+x*v-y*u;
```

Let us restrict our attention to the right hemisphere of  $S^3$ . So

```
> c:=+1;
```

Define, for lack of a better name:

```
> den:=1+x^2+y^2+z^2;
```

Then

```
> alpha:=(1/den)*sqrt(K*P^2+Q^2+R^2);
> beta:=(1/den)*sqrt(K-1)*P;
```

Here, we have simply chosen the + sign in the drift term  $\beta$ . A moment's thought shows that there is no loss of generality in doing so. The Finsler function  $F$  and its associated Lagrangian  $L$  are:

```
> F:=alpha+beta;
> L:=(1/2)*F^2;
```

Use of the semi-colon instructs Maple to display the input formulas.

### A.2. The covariant form of the geodesic spray coefficients.

By that, we are referring to the quantities

$$G_i := \frac{1}{2} \gamma_{ijk} y^j y^k = \frac{1}{4} (g_{ij,x^k} - g_{jk,x^i} + g_{ki,x^j}) y^j y^k.$$

Raising the index with the inverse of the fundamental tensor  $g_{ij}$  gives the contravariant form  $G^i$ . That will be carried out in §A.4. In terms of the Lagrangian  $L$ , it is not difficult to show that

$$G_i = \frac{1}{2} (L_{y^i x^j} y^j - L_{x^i}).$$

Our natural coordinates  $x^1, x^2, x^3$  are denoted  $x, y, z$ , and the induced tangent space coordinates  $y^1, y^2, y^3$  are denoted  $u, v, w$ . The covariant form of the geodesic spray coefficients are named (by us)  $g1, g2, g3$  in Maple. Thus

```
> Lx:=diff(L,x);
> Ly:=diff(L,y);
> Lz:=diff(L,z);
> g1:=(1/2)*(u*diff(Lx,u)+v*diff(Ly,u)+w*diff(Lz,u)-Lx);
> g2:=(1/2)*(u*diff(Lx,v)+v*diff(Ly,v)+w*diff(Lz,v)-Ly);
> g3:=(1/2)*(u*diff(Lx,w)+v*diff(Ly,w)+w*diff(Lz,w)-Lz);
```

Use of the colon instructs Maple to suppress the computed answers.

### A.3. The inverse of the fundamental tensor.

The inverse  $g^{ij}$  of the fundamental tensor  $g_{ij}$  (§2) is needed in order to obtain the contravariant form  $G^i$  of the geodesic spray coefficients. This inverse has a standard formula

$$g^{ij} = \frac{\alpha}{F} \tilde{a}^{ij} + \frac{\alpha^2}{F^2} \frac{\beta + \alpha \|\tilde{b}\|^2}{F} \tilde{\ell}^i \tilde{\ell}^j - \frac{\alpha^2}{F^2} \left( \tilde{\ell}^i \tilde{b}^j + \tilde{\ell}^j \tilde{b}^i \right),$$

where  $\tilde{\ell}^i := \frac{y^i}{\alpha}$ ,  $\tilde{b}^i := \tilde{a}^{ij} \tilde{b}_j$ , and  $\tilde{a}^{ij}$  is the inverse of the Riemannian metric  $\tilde{a}_{ij}$ . For a pedagogical derivation, one can consult [BCS].

In our natural coordinates, the matrix of the Riemannian metric  $\tilde{a}$  is

$$\frac{1}{\text{den}^2} \begin{pmatrix} K + z^2 + y^2 & -Kcz + cz - xy & Kcy - xz - cy \\ -Kcz + cz - xy & Kz^2 + 1 + x^2 & -Kyz \\ Kcy - xz - cy & -Kyz & Ky^2 + x^2 + 1 \end{pmatrix},$$

with  $den := 1+x^2+y^2+z^2$ . The inverse of  $\tilde{a}_{ij}$  has been computed elsewhere using Maple. We assign it the name  $A_{ij}$  in our codes:

```

> A11:=(1/K)*((x^2+1)*(x^2+K*z^2+K*y^2+1)):
> A12:=(1/K)*(x^3*y+x*y^3*K+x*y-c*z*x^2-c*z
  +K*c*z*x^2+K*c*z+K*y*z^2*x):
> A13:=(1/K)*(K*y^2*z*x+x^3*z+x*y^3*K+x*z
  -K*c*y*x^2-K*c*y+c*y*x^2+c*y):
> A21:=A12:
> A22:=(1/K)*(y^2*x^2+K*x^2+(K-1)*2*x*y*c*z
  +y^4*K+z^2+K+y^2*K*z^2+2*K*y^2):
> A23:=(1/K)*(x^2*z*y+(K-1)*c*z^2*x-(K-1)*x*y^2*c
  +K*y^3*z+K*y*z^3-z*y+2*y*K*z):
> A31:=A13:
> A32:=A23:
> A33:=(1/K)*(z^2*x^2+K*x^2-(K-1)*2*x*y*c*z
  +y^2*K*z^2+z^4*K+2*K*z^2+K+y^2):

```

As we have previously decided,  $c = +1$  because we want to focus on the right hemisphere. The (covariant) components  $\tilde{b}_i$  of our drift 1-form are

$$\frac{\sqrt{K-1}}{1+x^2+y^2+z^2} (c, -z, +y).$$

Denote its contravariant components  $\tilde{b}^i := \tilde{a}^{ij}\tilde{b}_j$  in Maple as  $B1$ ,  $B2$ ,  $B3$ .

On the right hemisphere:

```

> kappa:=sqrt(K-1)/den:
> B1:=kappa*(A11-z*A12+y*A13):
> B2:=kappa*(A21-z*A22+y*A23):
> B3:=kappa*(A31-z*A32+y*A33):

```

In Maple, let us denote  $\tilde{\ell}^i := \frac{y^i}{\alpha}$  as  $tel1$ ,  $tel2$ ,  $tel3$ , and the Riemannian norm

$$\|\tilde{b}\| = \sqrt{\frac{K-1}{K}} = \sqrt{1 - \frac{1}{K}} \text{ as } B.$$

We have:

```

> tel1:=u/alpha:
> tel2:=v/alpha:
> tel3:=w/alpha:
> B:=sqrt(1-(1/K)):

```

To reduce clutter, let us also introduce two abbreviations:

```

> rho:=alpha/F:
> phi:=(beta+alpha*B^2)/F:

```

Now we are ready to give the formula for the inverse  $g^{ij}$  of the fundamental tensor. Denote that inverse, in Maple, as  $G_{ij}$ . Then:

```

> G11:=rho*A11+rho^2*phi*tel1*tel1-rho^2*(tel1*B1+tel1*B1):
> G12:=rho*A12+rho^2*phi*tel1*tel2-rho^2*(tel1*B2+tel2*B1):
> G13:=rho*A13+rho^2*phi*tel1*tel3-rho^2*(tel1*B3+tel3*B1):
> G21:=G12:
> G22:=rho*A22+rho^2*phi*tel2*tel2-rho^2*(tel2*B2+tel2*B2):
> G23:=rho*A23+rho^2*phi*tel2*tel3-rho^2*(tel2*B3+tel3*B2):
> G31:=G13:
> G32:=G23:
> G33:=rho*A33+rho^2*phi*tel3*tel3-rho^2*(tel3*B3+tel3*B3):

```

#### A.4. The contravariant form of the geodesic spray coefficients.

These are obtained by taking the covariant form  $G_i$  of the coefficients and raising the index with the inverse  $g^{ij}$  of the fundamental tensor.

```

> G1:=G11*g1+G12*g2+G13*g3:
> G2:=G21*g1+G22*g2+G23*g3:
> G3:=G31*g1+G32*g2+G33*g3:

```

#### A.5. Getting set up for Berwald's formula.

We assign names to the first and second order partial derivatives of  $G^i$ . This will avoid having to re-compute them every time they are needed. Our Maple codes for Berwald's formula should run more efficiently as a result of this move.

```

> G1x:=diff(G1,x):
> G1ux:=diff(G1u,x):
> G1vx:=diff(G1v,x):
> G1wx:=diff(G1w,x):

```

Duplicate the above with  $x$  replaced by  $y, z, u, v, w$ .

Then duplicate all with  $G1$  replaced successively by  $G2$  and  $G3$ .

#### A.6. The spray curvature à la Berwald's formula.

We now get Maple to calculate the nine components  $K^i_k$  of the spray curvature using Berwald's formula

$$K^i_k = 2(G^i)_{x^k} - y^j(G^i)_{x^j y^k} - (G^i)_{y^j}(G^j)_{y^k} + 2G^j(G^i)_{y^j y^k}.$$

Even though the covariant form  $K_{ik}$  of the spray curvature is symmetric in its two indices, the same *cannot* usually be said of the type  $(1^1)$  form  $K^i_k := g^{ij}K_{jk}$ . So, even at a purely numerical level,  $K^k_i \neq K^i_k$  in general, unless of course one is using a  $g$ -orthonormal frame.

Let us use  $Kay[i, k]$  as the Maple names for those nine spray curvatures. The reason for not using  $K[i, k]$  is because of the next command.

```
> Kay:=array(1..3,1..3):
```

Had we used  $K[i, k]$ , the above would read “ $K:=array(1..3,1..3):$ ”, which would wreak havoc with our codes since the Maple variable  $K$  already stands for something else (namely the constant positive flag curvature).

```

> Kay[1,1]:=2*G1x
  -G1u*G1u-G1v*G2u-G1w*G3u
  -u*G1ux-v*G1uy-w*G1uz
  +2*G1*G1uu+2*G2*G1uv+2*G3*G1uw:
> Kay[1,2]:=2*G1y
  -G1u*G1v-G1v*G2v-G1w*G3v
  -u*G1vx-v*G1vy-w*G1vz
  +2*G1*G1vu+2*G2*G1vv+2*G3*G1vw:
> Kay[1,3]:=2*G1z
  -G1u*G1w-G1v*G2w-G1w*G3w
  -u*G1wx-v*G1wy-w*G1wz
  +2*G1*G1wu+2*G2*G1wv+2*G3*G1ww:
> Kay[2,1]:=2*G2x
  -G2u*G1u-G2v*G2u-G2w*G3u
  -u*G2ux-v*G2uy-w*G2uz
  +2*G1*G2uu+2*G2*G2uv+2*G3*G2uw:
> Kay[2,2]:=2*G2y
  -G2u*G1v-G2v*G2v-G2w*G3v
  -u*G2vx-v*G2vy-w*G2vz
  +2*G1*G2vu+2*G2*G2vv+2*G3*G2vw:
> Kay[2,3]:=2*G2z
  -G2u*G1w-G2v*G2w-G2w*G3w
  -u*G2wx-v*G2wy-w*G2wz
  +2*G1*G2wu+2*G2*G2wv+2*G3*G2ww:
> Kay[3,1]:=2*G3x
  -G3u*G1u-G3v*G2u-G3w*G3u
  -u*G3ux-v*G3uy-w*G3uz
  +2*G1*G3uu+2*G2*G3uv+2*G3*G3uw:
> Kay[3,2]:=2*G3y
  -G3u*G1v-G3v*G2v-G3w*G3v
  -u*G3vx-v*G3vy-w*G3vz
  +2*G1*G3vu+2*G2*G3vv+2*G3*G3vw:
> Kay[3,3]:=2*G3z
  -G3u*G1w-G3v*G2w-G3w*G3w
  -u*G3wx-v*G3wy-w*G3wz
  +2*G1*G3wu+2*G2*G3wv+2*G3*G3ww:
```

#### A.7. The criterion for having constant flag curvature.

Finally, we ask Maple to check whether our Randers metric  $F$  has constant flag curvature  $K$ . The criterion we shall use has been derived in §6. It reads

$$K^i_k = K F^2 \left( \delta^i_k - \frac{y^i}{F} F_{y^k} \right) =: K \tau^i_k,$$

where  $\tau^i_k$  equals  $F^2$  times the terms inside the parentheses. The pertinent Maple codes are:

```
> Fu:=diff(F,u);
> Fv:=diff(F,v);
> Fw:=diff(F,w);
> tau:=array(1..3,1..3);
> tau[1,1]:=F^2-u*F*Fu;
> tau[1,2]:=-u*F*Fv;
> tau[1,3]:=-u*F*Fw;
> tau[2,1]:=-v*F*Fu;
> tau[2,2]:=F^2-v*F*Fv;
> tau[2,3]:=-v*F*Fw;
> tau[3,1]:=-w*F*Fu;
> tau[3,2]:=-w*F*Fv;
> tau[3,3]:=F^2-w*F*Fw;
```

The condition we are striving for is  $K^i_k = K \tau^i_k$ . To see if that's the case, we form the difference of the two sides, and also their ratio for good measure.

```
> with(linalg);
> f:= (i,k) -> Kay[i,k]-K*tau[i,k];
> dif:=matrix(3,3,f);
```

Hopefully the answers are zero.

```
> h:= (i,k) -> Kay[i,k]/(K*tau[i,k]);
> quot:=matrix(3,3,h);
```

These quotients should all be 1.

#### A.8. The verdict.

It was no problem for Maple to calculate *symbolically* those differences and quotients defined in §A.7. However, attempts to get Maple to symbolically simplify the answers to a 0 or a 1 consistently crashed because the resulting expressions were too large.

Given that, we did the next best thing. We randomly selected numerical values for the position coordinates  $x, y, z$ , the velocity variables  $u, v, w$ , and the positive flag curvature  $K$ . Then we asked Maple to evaluate those differences and quotients at the stipulated  $x, y, z, u, v, w$ , and  $K$ . In retrospect, it now appears that for the purpose of comparing  $K^i_k$  to  $K \tau^i_k$ , forming the ratio of the two terms works better than taking their difference.

We noticed that the amount of RAM used in our computations is routinely in excess of 1 Gigabyte. It is not clear whether this is attributable to any inefficiency in our codes. A sampling of our numerical results are as follows:

```

> simplify(eval(diff[1,1],[x=1.0,y=2.0,z=-3.0,
u=3.1416,v=2.78,w=137.0,K=29.0]));
.009000000000
> factor(simplify(eval(quot[1,1],[x=1.0,y=2.0,z=-3.0,
u=3.1416,v=2.78,w=137.0,K=29.0])));
1.000000007
> simplify(eval(diff[1,2],[x=9.0,y=7.0,z=-5.0,
u=3.1416,v=2.78,w=137.0,K=31.0]));
.008218000000
> factor(simplify(eval(quot[1,2],[x=9.0,y=7.0,z=-5.0,
u=3.1416,v=2.78,w=137.0,K=31.0])));
.9999957122
> simplify(eval(diff[1,3],[x=9,y=7,z=-5,
u=31416,v=278,w=137,K=31]));
0
> factor(simplify(eval(quot[1,3],[x=9,y=7,z=-5,
u=31416,v=278,w=137,K=31])));
1
> simplify(eval(diff[2,1],[x=131,y=17,z=-59,
u=61413,v=872,w=1/137,K=2]));
0
> factor(simplify(eval(quot[2,1],[x=131,y=17,z=-59,
u=61413,v=872,w=1/137,K=2])));
1
> simplify(eval(diff[2,2],[x=1,y=2,z=-3,
u=71,v=5,w=1/137,K=29]));
0
> factor(simplify(eval(quot[2,2],[x=1,y=2,z=-3,
u=71,v=5,w=1/137,K=29])));
1
> simplify(eval(diff[2,3],[x=1.0,y=2.0,z=-3.0,
u=3.1416,v=2.78,w=137.0,K=29.0]));
-.005050000000
> factor(simplify(eval(quot[2,3],[x=1.0,y=2.0,z=-3.0,
u=3.1416,v=2.78,w=137.0,K=29.0])));
1.000000215

```

```

> simplify(eval(dif[3,1],[x=1,y=2,z=3,
u=5,v=7,w=11,K=13]));
0
> factor(simplify(eval(quot[3,1],[x=1,y=2,z=3,
u=5,v=7,w=11,K=13])));
1
> simplify(eval(dif[3,2],[x=199.7,y=-2.4168,z=3.5,
u=59,v=79,w=119,K=357]));
.00003253970835
> factor(simplify(eval(quot[3,2],[x=199.7,y=-2.4168,z=3.5,
u=59,v=79,w=119,K=357])));
.9999963000
> simplify(eval(dif[3,3],[x=1,y=2,z=-3,
u=71,v=5,w=1/137,K=29]));
0
> factor(simplify(eval(quot[3,3],[x=1,y=2,z=-3,
u=71,v=5,w=1/137,K=29]));
1

```

As one can see, the numerical evidence is overwhelmingly in favor of our Randers metric having constant positive flag curvature  $K$ .

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